The Reliability of Multistate, Multioutput Systems

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A system which can generate multiple signals such that each signal can have different quality states is modelled. A measure of the effectiveness of the system as a function of the status of its components is developed. Assuming only that the probability distribution of the status of each component is known, bounds on the probability distribution of the system's measure of effectiveness are developed.

I. Introduction

In the modelling of systems of many components to study their reliability, the usual assumption has been that each component can be in one of two states, either operating or failed, and that the system itself exhibits the same behavior. In many cases, and in particular in the case of NASA's Deep Space Network (DSN) these assumptions are too simplistic. For example, in the frequency and timing system (FTS) of a Deep Space Station (DSS) many different time signals and frequency standards are output, and it is quite possible that some of these may cease to be generated, due to equipment failure for example, while others are still available. In this situation, one cannot state that the system is failed or that it is operating, but rather that it is operating in a degraded mode, providing only some of the services it is intended to provide. One of the objectives of this article is to develop methodology to quantify the performance of systems with this characteristic.

A second complication is introduced by the fact that each of the outputs of the system may be available in different qualities as the system degrades and is repaired. For example in the FTS, the signals can originate from hydrogen masers, cesium standards, rubidium standards or crystal oscillators, and the quality of each signal (accuracy and stability) will depend on which type of source is being used to generate it. Furthermore, as each signal is processed by the other components of the system, it may also be degraded in quality, depending on the state of the component. These varying qualities must be considered in the description of the system, since they influence the possible uses of the system's output. This article presents a method for the characterization of a system which can produce multiple outputs at multiple quality levels. The outputs are produced and processed by the components of the system.

We will assume that the components may be in different states and that these states can be assigned numerical values, with 0 representing a failed component. We will also assume that the state is a measure of the quality of the signal that the component can put out, if a signal of that quality or better is available as input to the component. Thus, we do not allow for a component to produce an output of better quality than its input.

In Section II we present a method of describing the quality of a single output as a function of the state of the components. Normally we are interested in predicting the system's future behavior, and the state of the components will not be known with certainty. We thus proceed to assume that the states of the components are not precisely determined, but that we can give a probability distribution for the state of each component and determine from these distributions the probability distribution of the quality level of the output. Since the computations involved are easily seen to be quite elaborate, we also present upper and lower bounds on the distribution of the state of the output, which are easier to compute.

In Section III we extend these results to the case of multiple outputs by considering the different activities (e.g., telemetry, navigation, radio science) that require the signals output by the system as well as the minimal quality of each signal that each of these activities requires. It is thus possible to determine, from the state of the components, whether all the signals required by an activity are available at the necessary quality level so that the activity may be carried out. By assigning a value to each activity, it is then possible to arrive at an overall measure of the operation of the system as a whole. We develop bounds on the probability that each activity can be carried out, since the exact computation in many cases is not practical.

In Section IV we present an application of the method to a portion of the FTS.

II. Analysis of a Single Output Multistate System

For a single signal, we define a *path set* as a set of components whose functioning will ensure that the signal is being produced. A *minimal path set* is a path that does not contain other paths. Thus, if all components are failed except those on a minimal path, the signal will be produced, but if any one of the minimal path components subsequently fails, the signal will cease to be produced.

We also define a *cut set* as a set of components whose failure causes the signal to cease to be produced, even if all other components are functioning. A *minimal cut set* is a cut set that does not contain other cut sets. Thus, if all components are functioning except those on a minimal cut set, the signal will not be produced, but repair of a single component of the minimal cut will cause the signal to be produced again.

We remark that the concepts of path and cut are of a binary nature and do not depend on the actual states of the components, except for the failed not failed distinction. We will initially consider only systems in which components can be in one of two states, and later extend our results to the more general case of multistate components. Corresponding to this binary conception, we may define the indicator variables x_i as

$$x_i = \begin{cases} 1 \text{ if component } i \text{ is functioning} \\ 0 \text{ if component } i \text{ is not functioning} \end{cases}$$

and the signal indicator variable ϕ by

$$\phi = \begin{cases} 1 \text{ if the signal is being produced} \\ 0 \text{ if the signal is not being produced} \end{cases}$$

Clearly ϕ depends only on the values of x_i, $i = 1, 2 \ldots n$, where n is the number of components that generate the signal. We thus write $\phi = \phi(x_i, x_2, \ldots, x_n) \equiv \phi(\underset{\sim}{x})$. As discussed in Barlow and Proschan (Ref. 1), if $\mathscr{P}_i$, $i = 1, 2, \ldots, p$ are the min path sets, and $\mathscr{K}_i$, $i = 1, 2, \ldots k$ are the min cut sets, then we can write

$$\phi(\underset{\sim}{x}) = \prod_{i=1}^{k} \left[1 - \prod_{j \in \mathscr{K}_i} (1 - x_j)\right] \equiv \min_{1 \leq i \leq k} \left\{\max_{j \in \mathscr{K}_i} x_j\right\}$$

$$= 1 - \prod_{i=1}^{p} \left(1 - \prod_{j \in \mathscr{P}_i} x_j\right) \equiv \max_{1 \leq i \leq p} \left\{\min_{j \in \mathscr{P}_i} x_j\right\}$$

As discussed in Ref. 1, these expressions can be expanded into multilinear expressions. Furthermore since the x_i are binary variables, $x_i^n = x_i$ for all n, so that no powers of the x_i appear.

If we now assume that components behave randomly, and let x_i be the state of component i, then ϕ is also a random variable. Under the assumption that the x_i are independent, and letting $p_i = P[x_i = 1] = E_{x_i}$, we can show that

$$P[\phi(\underset{\sim}{x}) = 1] = E\phi(\underset{\sim}{x})$$

is a function only of the p_i. This is known as the reliability of the signal and we write

$$E\phi(x) = h(p_1, p_2, \ldots, p_n) \equiv h(\underset{\sim}{P})$$

As explained in Ref. 1, $h(\underset{\sim}{P})$ can be obtained from the structure function by substituting for each x_i in the multilinear expansion of $\phi(\underset{\sim}{x})$ the corresponding p_i.

We remark that in the actual expansion of the structure ϕ and the reliability h, not all components x_i or p_i will be present, since some components of the system will not be relevant to the production of the signal. We thus distinguish between relevant and irrelevant components for the production of the signal.

We now proceed to extend the above results to the case where each component can be in any one of many states. Assume that the state of component j can be represented by a variable z_j. If we consider a minimal path of the signal, then, since all components in that path are necessary to process it, the best signal that path can produce is

$$\min_{j \in \mathscr{P}_i} z_j$$

Also, since any of the min paths is sufficient to generate the signal, the value of the actual signal produced will be

$$\Psi_p (z_1, z_2, \ldots, z_n) = \max_{1 \leq i \leq p} \left\{ \min_{j \in \mathscr{P}_i} z_j \right\}$$

Similarly, since the signal must go through at least one component in a minimal cut set, the value of that cut set will be

$$\max_{j \in \mathscr{K}_i} z_j$$

and since the signal must traverse all minimal cut sets, we are led to the alternative

$$\Psi_k (z_1, z_2, \ldots, z_n) = \min_{1 \leq i \leq k} \left\{ \min_{j \in \mathscr{K}_i} z_j \right\}$$

It is of course evident that if we restrict z_i to binary variables, we obtain precisely the structure ϕ that details whether the signal is being produced or not.

A property of our definitions which will allow us to use the power of the binary system theory of Ref. 1 is the following. Consider a binary structure ϕ with min path sets $\mathscr{P}_i$, $i = 1, 2, \ldots, p$, and min cut sets $\mathscr{K}_i$, $i = 1, 2, \ldots, k$. Assume component j operates for time t_j and then fails. Then the time until the structure fails is (See Ref. 1, pp. 12).

$$T = \min_{1 \leq i \leq k} \left\{ \max_{j \in \mathscr{K}_i} t_j \right\} = \max_{1 \leq i \leq p} \left\{ \min_{j \in \mathscr{P}_i} t_j \right\}$$

By comparing this expression with our definition of Ψ_p and Ψ_k, we conclude that they are both equal, and furthermore establish the following:

Theorem

The state of the system is equal to the life of a binary system with the same min paths and min cuts, and whose component lifetimes are equal to the state of the individual components.

We will call the binary structure of the theorem the equivalent binary structure. With this property, most of the results known for binary structures extend easily to this more general representation. In particular all results on modular decompositions are valid. We can now assign probabilities to the states of the components, and obtain probabilistic results for the state of the system. If $\overline{F}_i(z) = P\,[Z_i > z]$ then $\overline{F}(z) = P\,[\Psi(\underset{\sim}{Z}) > z]$ is equal to the probability that the life of the equivalent binary structure exceeds z is thus (Ref. 1)

$$\overline{F}(z) = h[\,\overline{F}_1(z), \ldots, \overline{F}_n(z)]$$

From this observation we immediately obtain versions of the IFRA and NBU closure theorems.

A distribution $F(x) = 1 - \overline{F}(x)$ is called increasing failure rate average (IFRA) if it has the property that $-1/x \log [1 - F(x)]$ is increasing in x. See Ref. 1, Section 4.2 for explanation and interpretation. The IFRA closure theorem states that a system whose components have IFRA distributions has itself an IFRA distribution (Ref. 1, Theorem 4.2.6).

If $-\dfrac{1}{z}\log \overline{F}_i(z)$ is increasing in z for each i, then

$-\dfrac{1}{z}\log \overline{F}(z)$ is increasing in z.

A distribution is called new better than used (NBU) if it has the property that for all x_1 and x_2

$$1 - F(x_1 + x_2) \leq [1 - F(x_1)]\;[1 - F(x_2)]\;.$$

See Ref 1, Section 6.2 for an explanation and interpretation. The NBU closure theorem is analogous to the IFRA closure theorem (Ref. 1, Theorem 6.5.1).

If $\overline{F}_i(z_1 + z_2) \leq \overline{F}_i(z_1)\,\overline{F}_i(z_2)$ for each i for all z_1, z_2, then

$\overline{F}(z_1 + z_2) \leq \overline{F}(z_1)\,\overline{F}(z_2)$ for all z_1 and z_2.

Another result that follows immediately is the reliability bounds of Ref. 1.

Theorem

Let $p_i = P(Z_i > z)$ for a fixed z. Then if the components are independent

$$\prod_{i=1}^{k}\left[1 - \prod_{j\in\mathcal{K}_i}(1-p_j)\right] \leq \bar{F}(z) \leq 1 - \prod_{i=1}^{p}\left(1 - \prod_{j\in\mathcal{P}_i} p_j\right)$$

and

$$\max_{1\leq i\leq p}\left\{\prod_{j\in\mathcal{P}_i} p_j\right\} \leq \bar{F}(z) \leq \min_{1\leq i\leq k}\left\{1 - \prod_{j\in\mathcal{K}_i}(1-p_j)\right\}$$

Neither bound dominates so that in practice it is necessary to compute both pairs and select the best lower and upper bounds. Using modular decompositions, we can improve the bounds, as done in Ref. 1, pp. 39-44.

The concepts introduced thus far are stationary, and we will find it convenient to introduce time-varying states. If $Z_i(t)$ is the state of component i at time t, we define

$$\bar{F}_i(t,z) = P[Z_i(t) > z]$$

Thus for fixed t, $\bar{F}_i(t,z)$ is the distribution of the state at time t. If T_{iz} is the first time at which the state of component i goes below z, then

$$P(T_{iz} > t) = P[Z_i(t) > z] = \bar{F}_i(t,z)$$

Thus for fixed z, $\bar{F}_i(t,z)$ is the distribution of the time until the components state first goes below z.

Therefore, for any t and z if the components are independent

$$\bar{F}(t,z) = h[\bar{F}_1(t,z), \ldots, \bar{F}_n(t,z)]$$

and, for any fixed z, we have IFRA and NBU closure theorems for the passage times.

III. Multiple Signals

A possible description of the state of a multisignal system would be the vector whose components describe the state of each signal. However, this has various drawbacks. The more serious one is of course that it becomes difficult to state whether a particular state of the system is in some sense better or worse than another state. Furthermore, the number of possible states grows explosively with the number of signals, thereby obscuring the benefits of any analysis. The solution we have chosen to this dilemma is to consider the use to which the signals are put. We will assume that the outputs of the system are input to several users, which in the case of the DSN will be thought of as activities (e.g., telemetry, radio science, etc.). Each activity is assumed to require all of the signals, with a specified minimal quality level for each. Let

m_{ij} = minimal acceptable level of signal i required by activity j

It should be noted that we accept the possibility that $m_{ij} = 0$, which allows for the fact that signal i may actually not be required by activity j. Our formulation is preferable only because it leads to simpler notation. Note that this transforms each activity into a binary variable once more. Furthermore, we attach a value v_j to activity j, and we then measure the performance level of the system by the sum of the values of all the activities that can be performed. Thus, for activity j, we define, as an indicator of whether or not we are receiving the value of the activity, the binary random variable

$$Y_j = \begin{cases} 1 & \text{if } \Psi_i(z_1, z_2 \ldots z_n) \geq m_{ij} \text{ for } i = 1, 2, \ldots, m \\ 0 & \text{otherwise} \end{cases}$$

where n is the number of components that produce the signals, and m is the number of signals. The application of the probabilistic notions to the experiment is complicated by the fact that the signals are not independent

$$P(Y_j = 1) = P[\Psi_i(\underline{Z}) \geq m_{ij}, i = 1, 2, \ldots, m]$$

and since all the Ψ_i are functions of the random vector $\underline{Z}$, they are not independent. We thus are forced to either consider their interdependence, or to develop bounds on the probabilities that are relatively simple to compute.

The computation of bounds is relatively straightforward.

Theorem

$$\prod_{i=1}^{m} P(\Psi_i \geq m_{ij}) \leq P(Y_j = 1) \leq 1 - \prod_{i=1}^{m}[1 - P(\Psi_i \geq m_{ij})]$$

$$P(Y_j = 1) \leq \min_i P(\Psi_i \geq m_{ij})$$

Proof

Letting

$$\tau_i = \begin{cases} 1 & \text{if } \Psi_i(\underset{\sim}{Z}) \geqslant m_{ij} \\ 0 & \text{otherwise} \end{cases}$$

We have that

$$Y_j = \begin{cases} 1 & \text{if } \tau_i = 1, i = 1, 2, \ldots, m \\ 0 & \text{otherwise} \end{cases}$$

Hence Y_j is a series system of the τ_i. Note that the τ_i are not independent since they all depend on the same set of variables. However, since they are non-decreasing functions of the independent random variables $\underset{\sim}{Z}$, they exhibit a special form of dependence known as association (see Ref. 1, Section 2.2 for definition and properties). Therefore, the reliability bounds for a series system of associated components (Ref. 1, Section 2.3) apply, yielding

$$\prod_{i=1}^{m} P(\tau_i = 1) \leqslant P(Y_j = 1) \leqslant 1 - \prod_{i=1}^{m} [1 - P(\tau_i = 1)]$$

which is the desired result.

The advantage of upper and lower bounds lies in that they provide an estimate of the error of each bound. We can use these approximations to compute bounds on the expectations of the value of all the activities. Since activity j is worth V_j we obtain

$$E \sum_{j=1}^{l} Y_j V_j = \sum_{j=1}^{l} P(Y_j = 1) V_j$$

and this can be bounded above and below by the bounds on each individual term.

The computation of the exact reliability can, at least in principle, be carried out along the following lines: Let $\phi_i(\underset{\sim}{x})$ be the binary structure of signal i. Let

$$\phi(\underset{\sim}{x}) = \prod_{i=1}^{m} \phi_i(\underset{\sim}{x})$$

be the structure of all activities (the activity can be performed if all signals are present). Although the product need only be taken over those signals that are actually used in activity j, the use of all signals will cause no problems once the minimal levels are introduced.

Corresponding to the structure ϕ defined above we have the reliability function

$$h(\underset{\sim}{P}) = P[\phi(\underset{\sim}{x}) = 1] = h(P_1, \ldots, P_n)$$

obtained as before by replacing x_i by p_i in the expansion of ϕ. We now let

$$M_{kj} = \max_i \{m_{ij}\}$$

(where the maximization is carried out only over those signals i in which component k is relevant.)

Therefore M_{kj} is the minimal level of operation of component k that is compatible with the operation of activity j. It follows that the probability that activity j can be carried out is

$$P[\Psi_j(\underset{\sim}{Z}) = 1]$$

$$= h[P(Z_1 \geqslant M_{1j}), P(Z_2 \geqslant M_{2j}), \ldots, P(Z_n \geqslant M_{nj})]$$

It should be evident that, even if the determination of $\phi_i(\underset{\sim}{x})$ could be carried out by analyzing every signal (and this is probably feasible from a practical point of view), the algebraic work necessary to obtain $\phi(\underset{\sim}{x})$ is probably beyond the realm of practicality due to the exponential growth in the number of terms (e.g., if each of 10 signals has 5 terms, a conservative estimate, we would have $5^{10} \sim 9.75$ million terms in the expansion of ϕ. This is what makes the possibility of using bounds so attractive. On the other hand, an alternative worth exploring is that of using symbol manipulation computer programs to expand and reduce these expressions, and this should be examined in the future.

IV. An Example

As an example, we will consider the generation of the 10.1-MHz frequency standard by the DSN. The block diagram (Fig. 1) is self-explanatory and is the basis of our analysis. From the block diagram we can easily deduce a logic diagram which specifies the binary structure of the system under consideration (see Fig. 2). By examining this diagram we can list the minimal path sets and minimal cut sets.

Minimal paths	Minimal cuts
1, 6, 7, 10	1, 3, 8
2, 3, 6, 7, 10	1, 3, 12
3, 4, 6, 7, 10	1, 2, 4, 5
3, 5, 6, 7, 10	1, 3, 4, 5
4, 6, 7, 8, 10, 12	7
5, 6, 7, 8, 10, 12	10
	6

Note that from the diagram, components 4 and 5 always appear in a parallel configuration and can thus be treated as a module, i.e., as a single larger component. Similarly, components 8 and 12 are in series and so are 6, 7 and 10.

Thus, if we let

$$X_A = X_4 + X_5 - X_4X_5$$

$$X_B = X_8X_{12}$$

$$X_C = X_6X_7X_{10}$$

we can rewrite the logic diagram as seen in Fig. 3. We obtain the following minimal path and minimal cuts

Minimal paths	Minimal cuts
$1, C$	$1, 3, B$
$2, 3, C$	$1, 2, A$
$3, A, C$	$1, 3, A$
B, A, C	C

Therefore, the structure function is (based on the minimal paths)

$$\phi(\underset{\sim}{X}) = 1 - (1 - X_1X_C)(1 - X_2X_3X_C)$$

$$(1 - X_3X_AX_C)(1 - X_AX_BX_C)$$

or, based on the minimal cut sets,

$$\phi(\underset{\sim}{X}) = [1 - (1 - X_1)(1 - X_3)(1 - X_B)]\ [1 - (1 - X_1)$$

$$(1 - X_2)(1 - X_A)]\ [1 - (1 - X_1)(1 - X_3)(1 - X_A)]X_C$$

Either expression can be reduced, after some painful algebra to

$$\phi(\underset{\sim}{X}) = X_1X_C + X_2X_3X_C + X_3X_AX_C + X_AX_BX_C$$

$$- (X_1X_2X_3X_C + X_1X_3X_AX_C + X_1X_AX_BX_C$$

$$+ X_2X_3X_AX_C + X_3X_AX_BX_C)$$

$$+ (X_1X_2X_3X_AX_C + X_1X_3X_AX_BX_C)$$

We emphasize that, even though this expression is easy to handle, we have not presented the algebra involved in obtaining it. Furthermore, the equivalent expression for a more complex system would be far harder both to obtain and to use. Therefore, the bounds developed will prove useful.

We now assume there are 5 qualities of signals:

0	No signal
6	Crystal standard quality
7	Rubidium standard quality
8	Cesium standard quality
9	Hydrogen maser standard quality

These numeric values were chosen as the logarithm of the Q value of typical devices of each kind. Any other assignment could, of course, be selected.

We will also assume known for some point in time t, values of the probabilities that different components are performing

Component 1:	$P[Z = 0] = 0.40$	$P[Z = 9] = 0.60$
Component 2:	$P[Z = 0] = 0.15$	$P[Z = 8] = 0.85$
Component 3:	$P[Z = 0] = 0.05$	$P[Z = 6] = 0.95$
Components 4 and 5:	$P[Z = 0] = 0.10$	$P[Z = 7] = 0.90$
Components 6-7-10:	$P[Z = 0] = 0.01$	$P[Z = 6] = 0.02$
	$P[Z = 7] = 0.02$	$P[Z = 8] = 0.05$
	$P[Z = 9] = 0.90$	
Components 8-12:	$P[Z = 0] = 0.01$	$P[Z = 7] = 0.99$

We can now compute

$$P_A = 0.9 + 0.9 - 0.9^2 = 0.99$$

and then proceed to compute the distribution of the state of the system. If we wanted $P\,[\Psi(\underset{\sim}{Z}) \geqslant 7]$ we would use

$$\begin{array}{llll} P_1 = 0.6 & & P_A = 0.99 \\ P_2 = 0.85 & & P_B = 0.99 \\ P_3 = 0 & & P_C = 0.97 \end{array}$$

to obtain

$$P\,[\Psi(Z) \geqslant 7] = 0.9623$$

The reliability lower bounds are

$$[1-(1-P_1)(1-P_3)(1-P_B)]\;[1-(1-P_1)(1-P_2)(1-P_A)]$$

$$[1-(1-P_1)(1-P_3)(1-P_A)]\;P_C = 0.9617$$

and

$$\max\,\{P_1P_C, P_2P_3P_C, P_3P_AP_C, P_BP_AP_C\} = 0.9507$$

Thus a lower bound is 0.9617

The reliability upper bounds are

$$1-(1-P_1P_C)(1-P_2P_3P_C)(1-P_2P_3P_C)$$

$$(1-P_AP_BP_C) = 0.9794$$

and

$$\min\,[1-(1-P_1)(1-P_3)(1-P_B), 1-(1-P_1)(1-P_2)(1-P_A),$$

$$1-(1-P_1)(1-P_3)(1-P_A), P_C] = 0.9700$$

Thus we obtain bounds of

$$0.9617 \leqslant P\,[\Psi(\underset{\sim}{Z}) \geqslant 7] \leqslant 0.9700$$

We can similarly compute bounds for the other states of the system

$0.9890 \leqslant P\,[\Psi(\underset{\sim}{Z}) \geqslant 6] \leqslant 0.9900$	(true value 0.9890)
$0.9617 \leqslant P\,[\Psi(\underset{\sim}{Z}) \geqslant 7] \leqslant 0.9700$	(true value 0.9623)
$0.5700 \leqslant P\,[\Psi(\underset{\sim}{Z}) \geqslant 8] \leqslant 0.5700$	(true value 0.5700)
$0.5400 \leqslant P\,[\Psi(\underset{\sim}{Z}) \geqslant 9] \leqslant 0.5400$	(true value 0.5400)

From these bounds it is possible to compute bounds on information such as the expected state of the system ($8.06 \leqslant E\,\Psi(\underset{\sim}{Z}) \leqslant 8.07$).

V. Summary

We have examined a system which can produce multiple outputs each of which can be of many different qualities. We presented a method of modelling such system as a function of the state of each of its components, and, when that state is known only as a probability distribution, we have shown how to determine the probability distribution of the measure of the system's effectiveness. It was shown that the exact computation of this distribution could be an impossible task and that therefore it may be attractive to have available upper and lower bounds whose computation is easier to carry out.

Acknowledgment

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Reference

1. Barlow, R. E., and Proschan, R., *Statistical Theory of Reliability and Life Testing*, Holt Rinehart Winston, 1975.

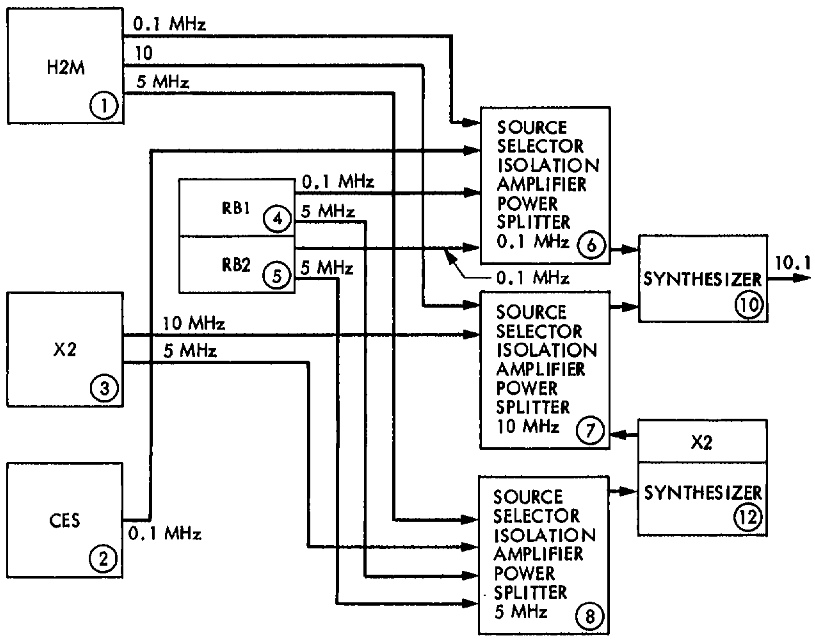

Fig. 1. Block diagram: 10.1 MHz frequency generation (numbers in circles are component numbers)

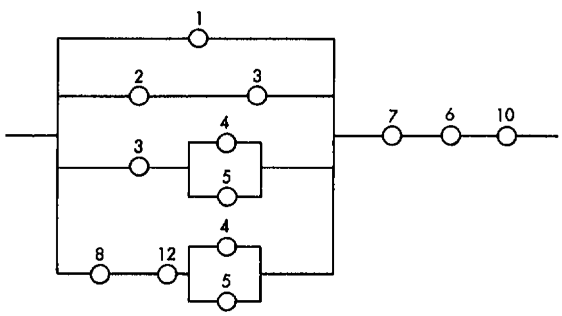

Fig. 2. Logic diagram

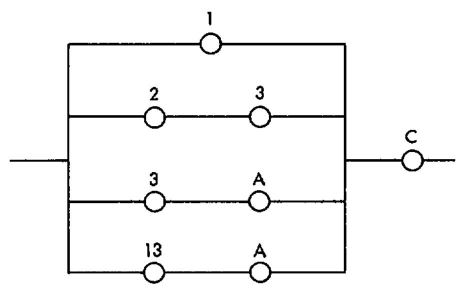

Fig. 3. Logic diagram including modules